Fundamentos de espectroscopia: Series de Fourier

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Expansión en series de Fourier

La serie de Fourier de la función $f(t)$ es una expansión en funciones seno y coseno :

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \operatorname{sen} \omega_n t \qquad (1)$$

- Cada término representa una oscilación a una frecuencia

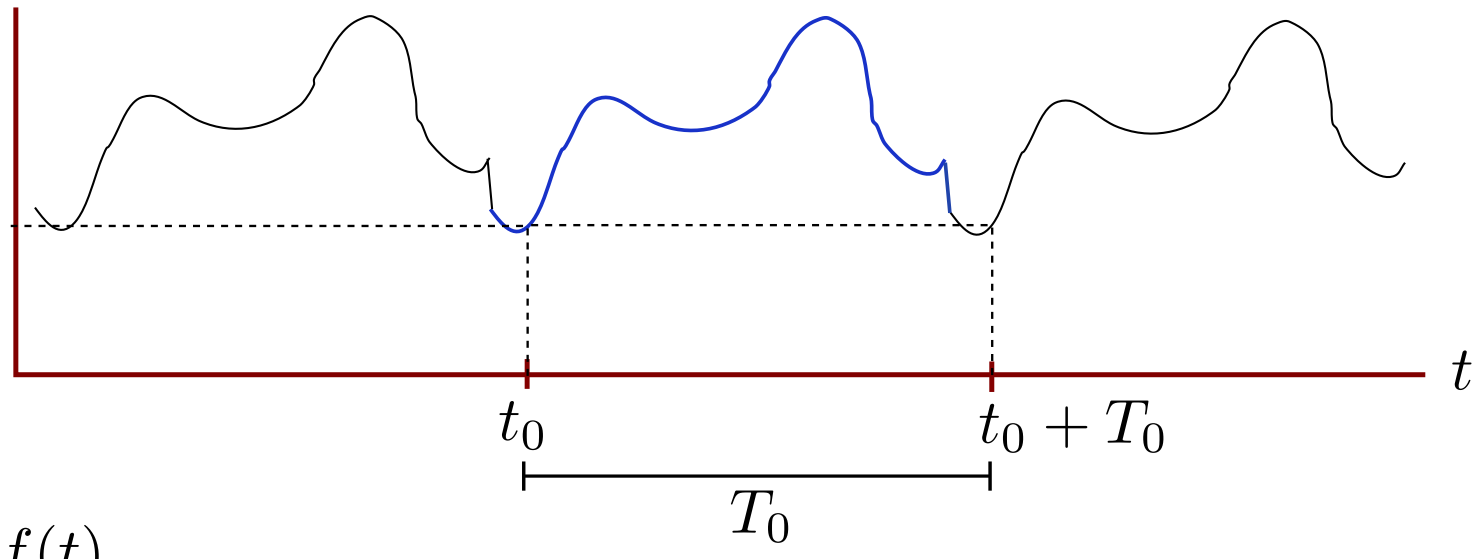
$$\omega_n = 2\pi n / T_0 \quad \omega_0 = 2\pi / T_0$$

- T_0: se determina por la periodicidad de $f(t)$ o por una periodicidad impuesta a alguna función.

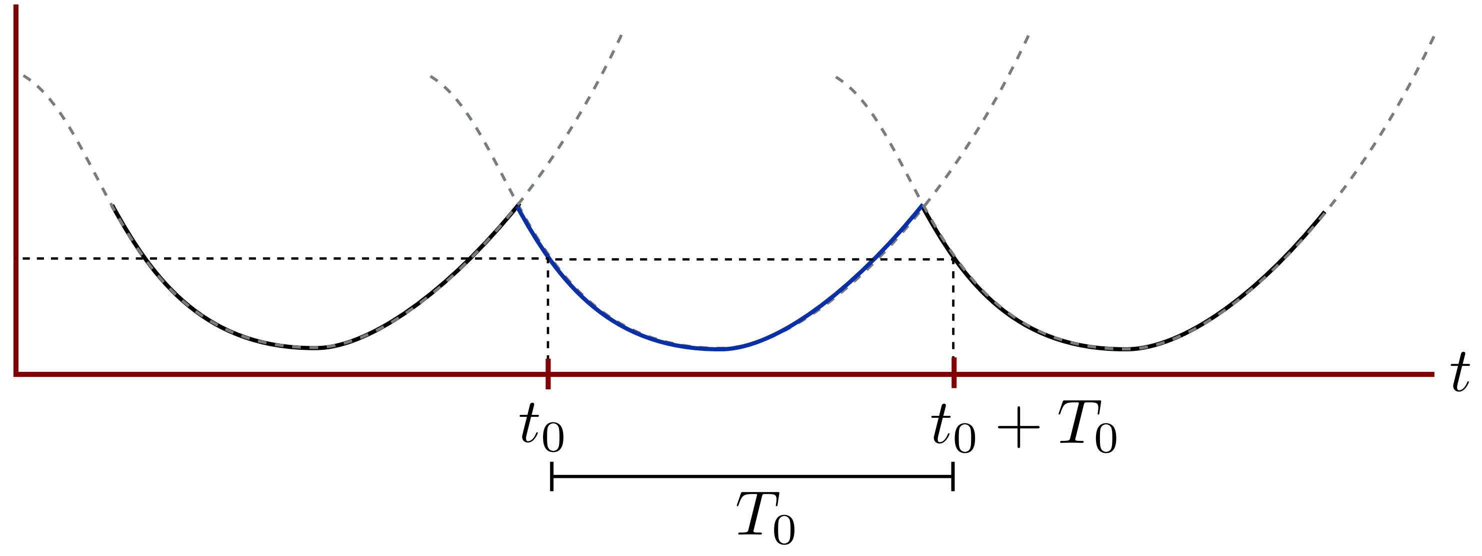
- Objetivo: Encontrar $\{a_n\}$ y $\{b_n\}$.

- Expansión en series de Fourier
- Apéndice

Caso 1: $f(t)$



Caso 2: $f(t)$



En ambos casos: $f(t + T_0) = f(t)$.

Obtención de a_0.

Integrar ambos lados de (1) entre t_0 y $t_o + T_0$, donde t_0 es un valor arbitrario:

$$\int_{t_0}^{t_0+T_0} f(t)\,dt = \int_{t_0}^{t_0+T_0} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\,\omega_n\,t + \sum_{n=1}^{\infty} b_n \,\mathrm{sen}\,\omega_n\,t\right] dt\,.$$

Debido a que

$$\int_{t_0}^{t_0+T_0} \mathrm{sen}(\omega_m t)dt = 0 \quad \text{y} \tag{2}$$

$$\int_{t_0}^{t_0+T_0} \cos(\omega_m t)dt = 0\,, \tag{3}$$

se obtiene:

$$a_0 = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} f(t)\,dt\,. \tag{4}$$

Obtención de a_n y b_n.

Para obtener a_n, multiplicar ambos lados de la ecuación (1) por $\cos(w_m x)$, $m > 0$, e integrar:

$$\int_{t_0}^{t_0+T_0} f(x)\ \cos(w_m t)\, dt = \int_{t_0}^{t_0+T_0} \left[\frac{a_0}{2}\ \cos(w_m t) + \sum_{n=1}^{\infty} a_n \cos(\omega_n t)\ \cos(w_m t) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(\omega_n t)\ \cos(w_m t)\right] dt\,. \quad (5)$$

Las funciones seno y coseno son ortogonales:

$$\int_{t_0}^{t_0+T_0} \operatorname{sen}(\omega_n t)\ \cos(w_m t)\, dt = 0 \quad (6)$$

$$\int_{t_0}^{t_0+T_0} \operatorname{sen}(\omega_n t)\ \operatorname{sen}(w_m t)\, dt = \delta_{mn}\frac{T_0}{2} \quad (7)$$

$$\int_{t_0}^{t_0+T_0} \cos(\omega_n t)\ \cos(w_m t)\, dt = \delta_{mn}\frac{T_0}{2} \quad (8)$$

Sustituir (3), (6) y (8) en (5):

$$\int_{t_0}^{t_0+T_0} f(t)\,\cos(w_m t)\,dt = \sum_{n=1}^{\infty} a_n \delta_{mn} \frac{T_0}{2}\,,$$

Sólo el término con $m \neq n$ es distinto de cero.

Por lo tanto:

$$a_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t)\,\cos(w_m t)\,dt\,. \tag{9}$$

De igual manera, se obtiene:

$$b_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t)\,\mathrm{sen}(w_m t)\,dt\,. \tag{10}$$

Resumen:

$$a_0 = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t)\, dt\,. \tag{4}$$

$$a_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t)\, \cos(w_m t)\, dt\,. \tag{9}$$

$$b_m = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t)\, \mathrm{sen}(w_m t)\, dt\,. \tag{10}$$

❖ Expansión en series de Fourier
❖ Apéndice

Ejemplo:

Realiza la expansión de la función $f(t) = t$ en series de Fourier,

$$f(t) = t = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \,\mathrm{sen}\, \omega_n t\,,$$

en el intervalo $t \in [-\pi, \pi]$.

- En este caso $t_0 = -\pi$ y $T_0 = 2\pi$.
- Por lo tanto: $\omega_0 = 2\pi/T_0 = 1$ y $\omega_n = n\,\omega_0 = n$.
- Con estos valores, se obtiene:

$$\begin{aligned} a_0 &= \frac{2}{T_0}\int_{-\pi}^{\pi} t dt = 0 \\ a_n &= \frac{2}{T_0}\int_{-\pi}^{\pi} t\cos(nt)dt = 0,\ n \geq 1 \\ b_n &= \frac{2}{T_0}\int_{-\pi}^{\pi} t\,\mathrm{sen}(nt)dt = -\frac{2(-1)^n}{n},\ n \geq 1 \end{aligned}$$

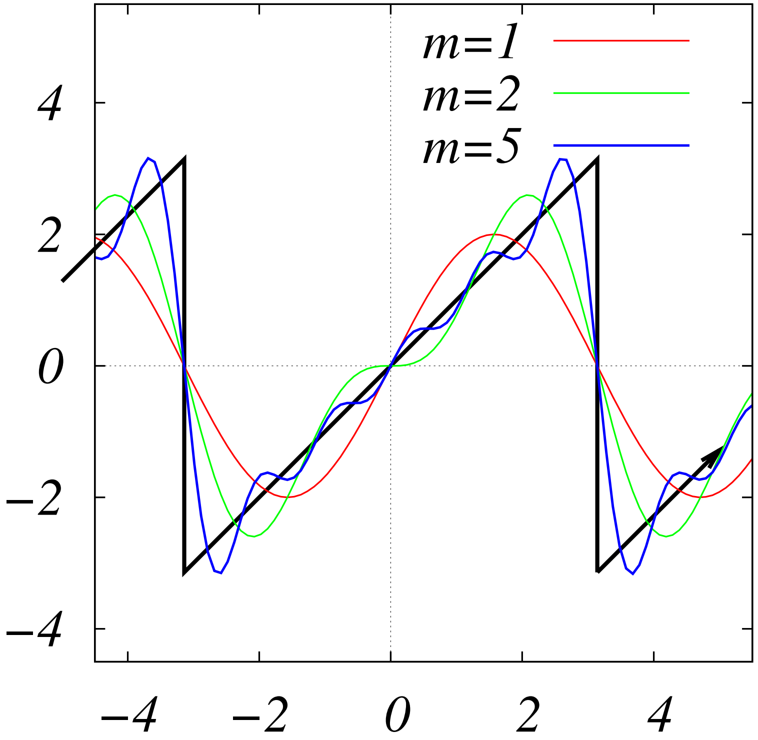
Estos valores se sustituyen en la definición de la serie:

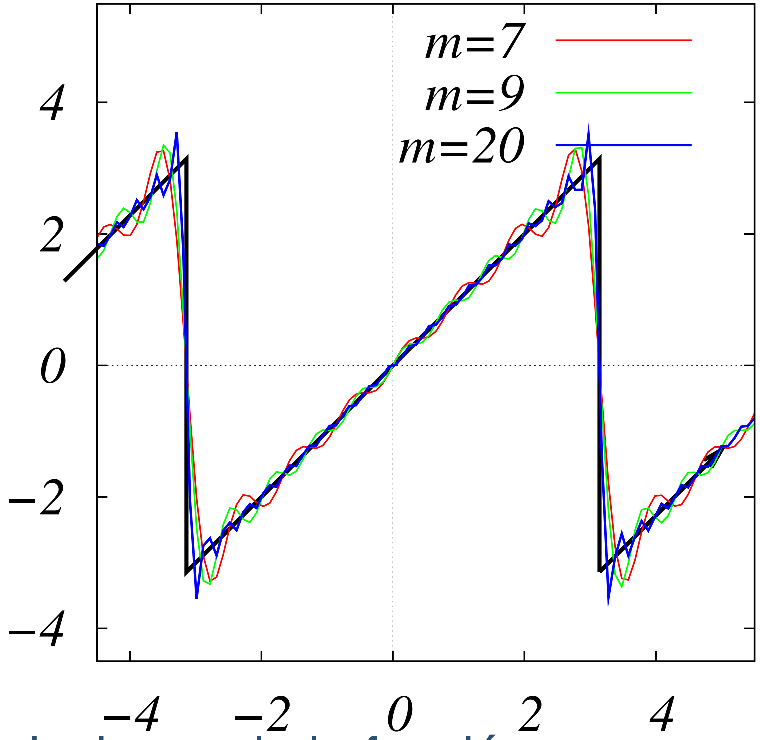
$$f(t) = t = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \operatorname{sen}(nt)$$

En la práctica, sólo se considera un número finito de términos, m, en la suma:

$$\begin{aligned} f(t) &= t \approx -\sum_{n=1}^{m} \frac{2(-1)^n}{n} \operatorname{sen}(nt) \\ &= 2 \operatorname{sen} t - \operatorname{sen} 2t + \frac{2}{3} \operatorname{sen} 3t - \frac{1}{2} \operatorname{sen} 4t + \ldots \end{aligned}$$

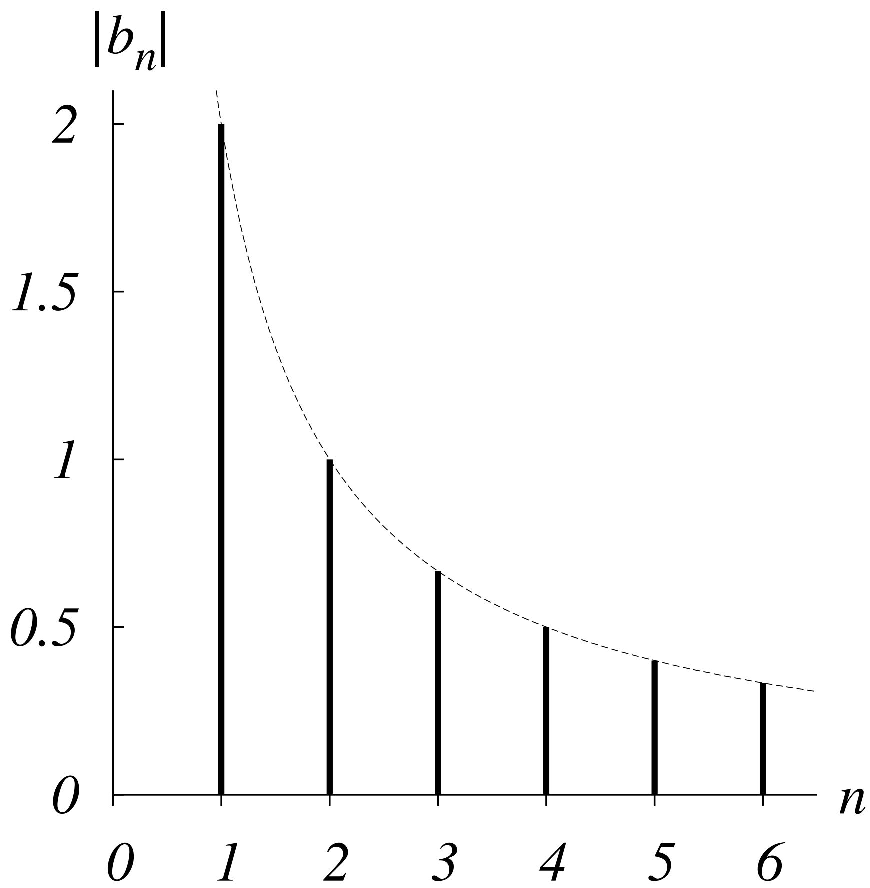
Algunos casos con diferentes valores de m para la expansión de la función en series de Fourier:





$\Rightarrow$ Fenómeno de Gibbs: grandes variaciones de la función aproximada alrededor de los puntos de discontinuidad.

El espectro es



Ejercicio:

Resolver el problema anterior con WOLFRAM MATHEMATICA

fourier-t.nb - Wolfram Mathematica 12.1

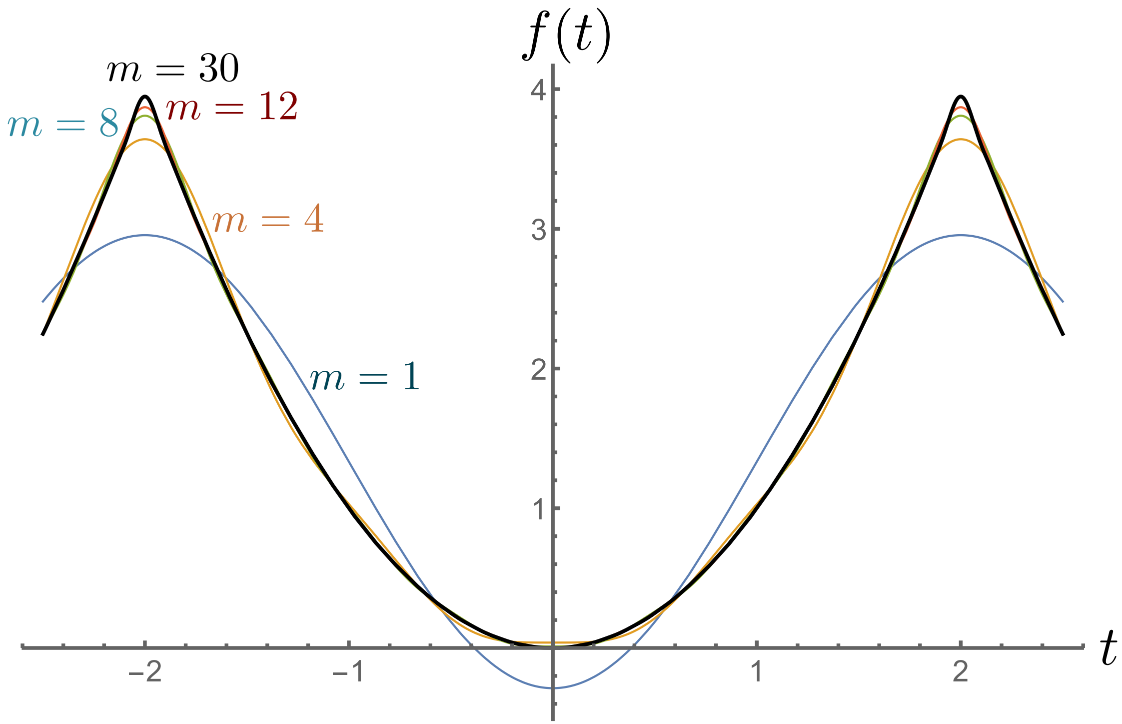
File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[•]:= ClearAll["Global`*"]

In[•]:= f[t_] := t

In[•]:= T0 = 2 Pi;
        t0 = -Pi;
        w0 = 2 Pi / T0;

In[•]:= a0 = (2 / T0) Integrate[f[t], {t, t0, t0 + T0}]

        (2 / T0) Assuming[n ∈ Integers,
          Integrate[f[t] Cos[n w0 t], {t, t0, t0 + T0}]]

In[•]:= (2 / T0) Assuming[n ∈ Integers,
          Integrate[f[t] Sin[n w0 t], {t, t0, t0 + T0}]]

In[•]:= b  = Table[- 2 (-1)^n / n, {n, 1, 40}]
```

```
In[•]:= b[[25]]

In[•]:= ListPlot[Abs[b], Filling -> Axis, PlotRange → Full]

In[•]:= g[t_, n_] := Sum[b[[m]] Sin[m w0 t], {m, 1, n}]

In[•]:= g[t, 3]

In[•]:= g[t, 1]

In[•]:= Plot[g[t, 1], {t, -5, 5}]

In[•]:= Manipulate[Plot[g[t, n], {t, -5, 5}], {n, 1, 40, 1}]
```

Ejercicio:

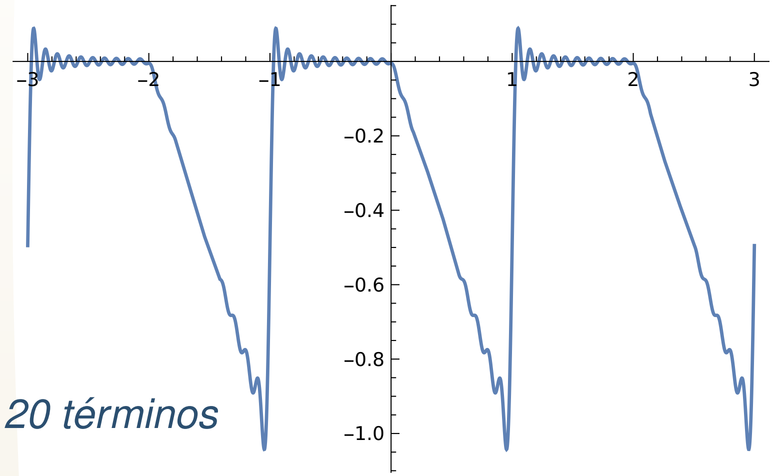
- Expresa la función $f(t) = t^2$ como serie de Fourier en el intervalo $t \in [-2, 2]$.



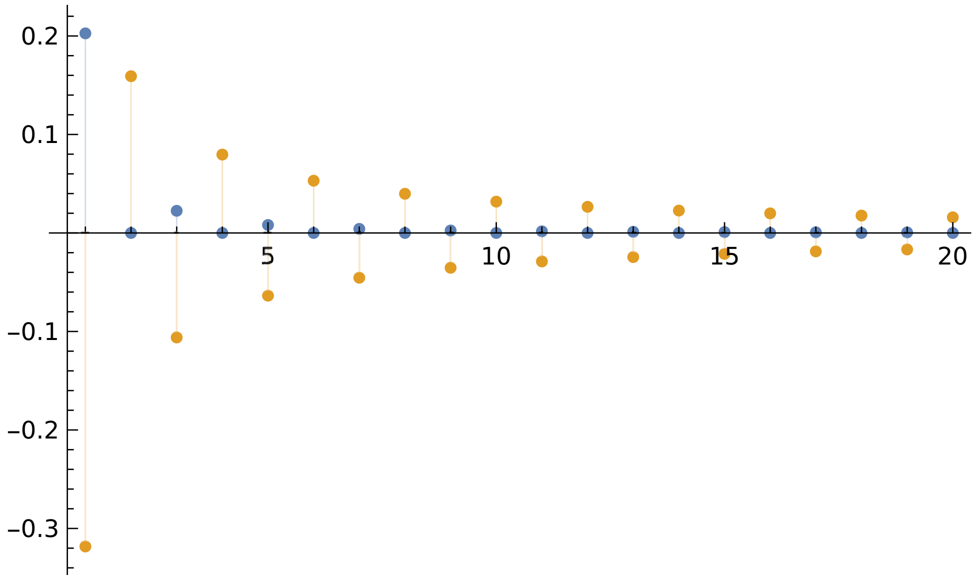
$$f(t) = t^2 \approx \frac{4}{3} + \sum_{n=1}^{m} \frac{(-1)^n\, 16}{\pi^2\, n^2} \cos(\omega_n t)$$

- Expresa la siguiente onda como serie de Fourier:

$$f(t) = \begin{cases} 0 & -1 \le t < 0 \\ -t & 0 \le t < 1 \end{cases}$$



Expansión con 20 términos



$$a_0 = -\frac{1}{2}$$

$$a_n = -\frac{(-1)^n - 1}{\pi^2 n^2}$$

$$b_n = \frac{(-1)^n}{\pi n}$$

Otros ejemplos:

Chapter 8 ■ Fourier Analysis **247**

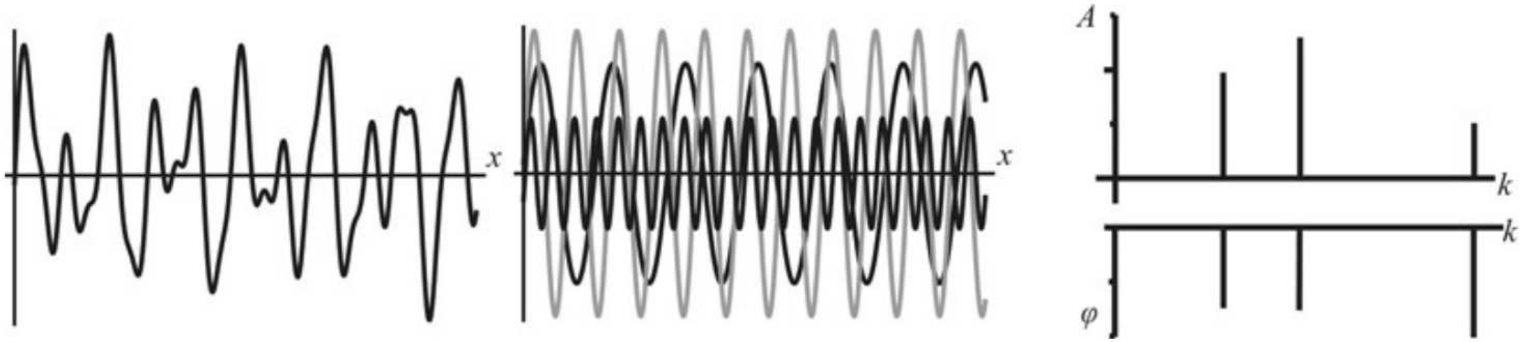


Figure 8.1.1 A seemingly complicated $y(x)$ (on the left) is actually just the sum of the three sinusoids shown in the center. The amplitude and phase of each sinusoid $A\cos(kx + \varphi)$ are shown on the right. (The amplitude is defined to be positive. For the three sinusoids here, all happen to have a negative phase.)

W Fox Smith, Waves and oscillations. A Prelude to Quantum Mechanics, Oxford University Press, 2010.

- ❖ Expansión en series de Fourier
- ❖ Apéndice

Otros ejemplos:

HARMONIC ANALYSIS AND SYNTHESIS

When a clarinet and an oboe play the same note, say, concert A, they sound quite different. Both notes have the same **pitch,** a physiological sensation of the highness or lowness of the note that is strongly correlated with frequency. However, the notes differ in what is called **tone quality.** The principal reason for the difference in tone quality is that, although both the clarinet and oboe are producing vibrations at the same fundamental frequency, each instrument is also producing harmonics whose relative intensities depend on the instrument and how it is played. If the sound produced by each instrument were entirely at the fundamental frequency of the instrument, they would sound identical.

Figure 16-25 shows plots of the pressure variations versus time for the sound from a tuning fork, a clarinet, and an oboe, each playing the same note. These patterns are called **waveforms.** The waveform for the sound from the tuning fork is nearly a pure sine wave, but those from the clarinet and the oboe are clearly more complex.

Waveforms can be analyzed in terms of the harmonics that constitute them by means of **harmonic analysis.** (Harmonic analysis is also called **Fourier analysis** after the French mathematician J.B.J. Fourier, who developed the

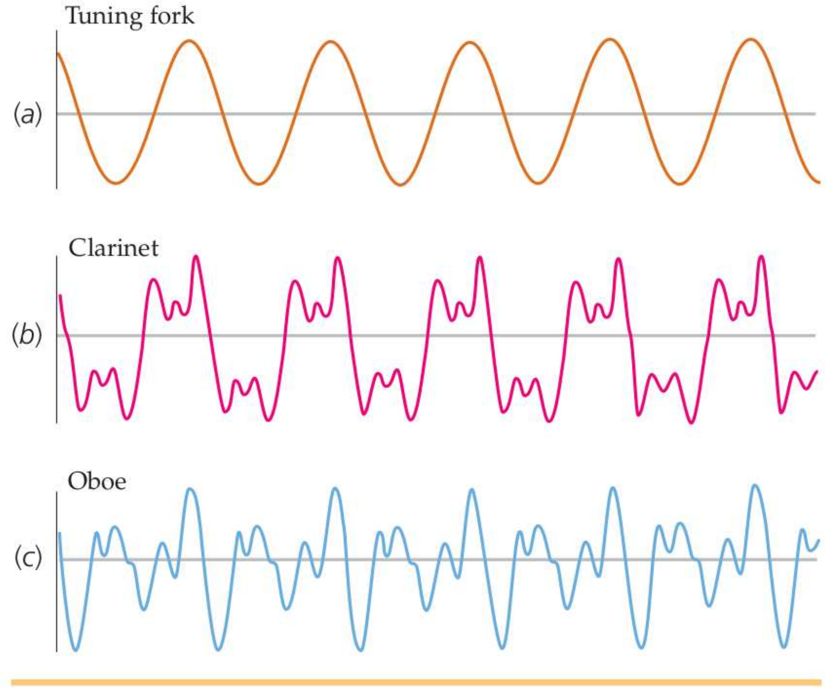


FIGURE 16-25 Waveforms of (*a*) a tuning fork, (*b*) a clarinet, and (*c*) an oboe, each at a fundamental frequency of 440 Hz and at approximately the same intensity.

Tipler & Mosca, *Physics for Scientists and engineers*, 6th edn, Freeman and Co., N. Y, 2008

Espectro de frecuencias:

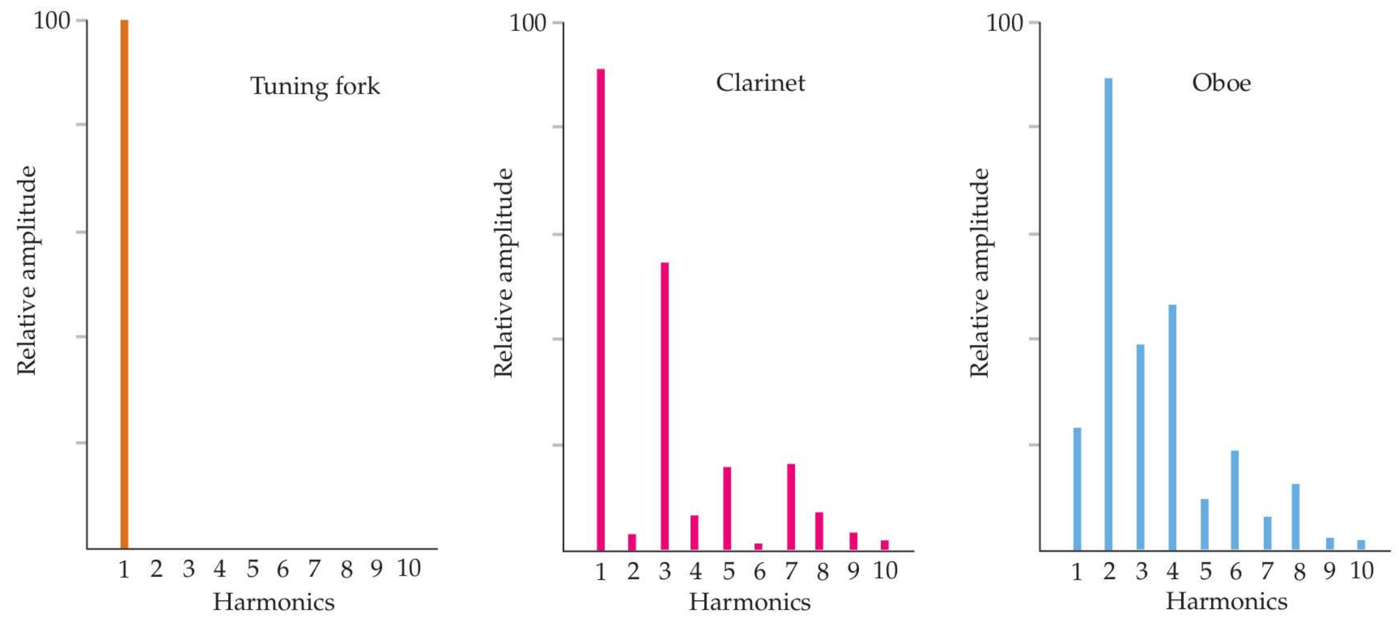


FIGURE 16-26 Relative intensities of the harmonics in the waveforms shown in Figure 16-25 for (*a*) the tuning fork, (*b*) the clarinet, and (*c*) the oboe.

Tipler & Mosca, *Physics for Scientists and engineers*, 6th edn., Freeman and Co., N. Y, 2008

https://toolster.net/tuning_fork

Apéndice

Expansión de una función f como combinación lineal de un conjunto de funciones base.

Sea $\{g_j\}$ una base ortogonal de un espacio de funciones:

$$\int g_i g_j d\tau = I_{ij}\delta_{ij}$$

donde

$$\delta_{ij} = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j \end{cases}$$

es la delta de Kronecker.

Expresar f en términos de $\{g_j\}$:

$$f = \sum_{j=1}^{n} k_j g_j$$

$\Rightarrow$ Hay que encontrar al conjunto $\{k_j\}$

$$f = \sum_{j=1}^{n} k_j g_j$$

Para encontrar k_i, multiplicar ambos lados por g_i e integrar:

$$g_i f = g_i \sum_{j=1}^{n} k_j g_j$$

$$\int g_i f d\tau = \sum_{j=1}^{n} k_j \int g_i g_j d\tau$$

$$f = \sum_{j=1}^{n} k_j g_j$$

Para encontrar k_i, multiplicar ambos lados por g_i e integrar:

$$g_i f = g_i \sum_{j=1}^{n} k_j g_j$$

$$\int g_i f d\tau = \sum_{j=1}^{n} k_j \int g_i g_j d\tau$$

$$= \sum_{j=1}^{n} k_j I_{ij} \delta_{ij}$$

$$f = \sum_{j=1}^{n} k_j g_j$$

Para encontrar k_i, multiplicar ambos lados por g_i e integrar:

$$g_i f = g_i \sum_{j=1}^{n} k_j g_j$$

$$\int g_i f d\tau = \sum_{j=1}^{n} k_j \int g_i g_j d\tau$$

$$= \sum_{j=1}^{n} k_j I_{ij} \delta_{ij} = k_i I_{ii}$$

$$f = \sum_{j=1}^{n} k_j g_j$$

Para encontrar k_i, multiplicar ambos lados por g_i e integrar:

$$g_i f = g_i \sum_{j=1}^{n} k_j g_j$$

$$\int g_i f d\tau = \sum_{j=1}^{n} k_j \int g_i g_j d\tau$$

$$= \sum_{j=1}^{n} k_j I_{ij} \delta_{ij} = k_i I_{ii}$$

Por lo tanto:

$$k_i = \frac{\int g_i f d\tau}{I_{ii}}$$

$\Rightarrow$ Si la base es ortonormal: $I_{ii} = 1$.